GROUP THEORY 2024 - 25, SOLUTION SHEET 2

Exercise 1. *Warm up exercise*

(1) Given en edge $\{i,j\} \in X$ one observes that there exists a permutation $\sigma \in S_4$ such that $\sigma \cdot \{1,2\} = \{i,j\}$. Therefore the orbit of $\{1,2\}$ is the set X itself.

(2) If a permutation in S_4 stabilises $\{1,2\}$ then it must permute the indices 1 and 2. Therefore we have

$$\mathrm{Stab}_{S_4}(\{1,2\}) = \{e,\ (1\,2), (3\,4), (1\,2)(3\,4)\}.$$

(3) Indeed,

$$|S_4 \cdot \{1,2\}||\mathrm{Stab}_{S_4}(\{1,2\})| = |6| \cdot |4| = 24 = |S_4|.$$

Exercise 2. *Action on cosets*

Denote the map $G \times G/H \to G/H$ by Φ. Denote by Φ_g the map

$$\begin{aligned} \Phi(g,-) : G/H &\to G/H \\ aH &\mapsto gaH \end{aligned}$$

(1)
- First, since we defined Φ_g using a specific representation aH of a coset, we need to show that Φ_g is well defined. We must show that for all $a, b \in G$ such that $aH = bH$, their images satisfy $\Phi_g(aH) = \Phi_g(bH)$. This holds because

$$\begin{aligned} \Phi_g(aH) &= (ga)H = \{(ga)h \mid h \in H\} \\ &= \{g(ah) \mid h \in H\} \\ &= \{g(bh') \mid h' \in H\} \\ &= \{(gb)h' \mid h' \in H\} \\ &= (gb)H = \phi_g(bH) \end{aligned}$$

  where we used the hypothesis $aH = bH$ on the third line.

- We observe that $\phi_g \circ \phi_{g^{-1}} = \mathrm{Id}_{G/H}$ and $\phi_{g^{-1}} \circ \phi_g = \mathrm{Id}_{G/H}$, which proves that Φ_g is bijective with inverse $(\phi_g)^{-1} = \phi_{g^{-1}}$, as desired.

- Finally, we prove that for all $g, g' \in G$, we have $\Phi_{gg'} = \Phi_g \circ \Phi_{g'}$. This is a straightforward computation: $\Phi_{gg'}(aH) = gg'aH = \Phi_g(g'aH) = (\Phi_g \circ \Phi_{g'})(aH)$.

(2) By definition,

$$\begin{aligned}\mathrm{Stab}_G(gH) &= \{g' \in G \mid \Phi_{g'}(gH) = gH\}\\ &= \{g' \in G \mid g'gH = gH\}\\ &= \{g' \in G \mid g^{-1}g'gH = H\}\\ &= \{g' \in G \mid g^{-1}g'g \in H\}\\ &= \{g' \in G \mid g' \in gHg^{-1}\} = gHg^{-1}.\end{aligned}$$

Exercise 3. Suppose first that K and H are conjugate subgroups of G with $K = gHg^{-1}$ for some fixed $g \in G$. We define a homomorphism of G - sets as follows:

$$\begin{aligned}\varphi : G/K &\to G/H\\ aK &\mapsto agH.\end{aligned}$$

Like in the previous exercise, we must show that φ is well defined. To this end, suppose that $aK = bK$. This is equivalent to saying that $b^{-1}a \in K$. We must show $agH = bgH$. This is equivalent to $g^{-1}b^{-1}ag \in H$, which holds since $b^{-1}a \in K$ and $g^{-1}Kg = H$. Moreover it is a morphism of G-sets since

$$\begin{aligned}\varphi(g' \cdot aK) &= \varphi(g'aK) = (g'ag)H\\ &= g' \cdot (agH) = g' \cdot \varphi(aK).\end{aligned}$$

We define now its inverse

$$\begin{aligned}\psi : G/H &\to G/K\\ aH &\mapsto ag^{-1}K.\end{aligned}$$

Using a similar argument, we can show that ψ is well defined and that it is a morphism of G-sets. It is clear that φ and ψ are inverse of each other, proving the claim.

Conversely, suppose that $\phi : G/H \to G/K$ is an isomorphism of G - sets. There exists $g \in G$ such that $\phi(H) = gK$. We aim to show that $K = gHg^{-1}$. Since ϕ is well defined, we have that

$$gK = \phi(H) = \phi(h^{-1}H) = h^{-1}\phi(H) = h^{-1}gK$$

for all $h \in H$. Hence $ghg^{-1} \in K$ for all $h \in H$ and so $gHg^{-1} \subseteq K$. On the other hand $\phi(H) = gK$ implies that $\phi^{-1}(K) = g^{-1}H$. Using the same argument for ϕ^{-1} and g^{-1} we obtain that $g^{-1}Kg \subseteq H$ which shows that $K = gHg^{-1}$. $\square$.

Exercise 4. (1) We define a map $\Phi : G \to Bij(X)$ by $g \mapsto \Phi_g$, where $\Phi_g(W) = g(W), \forall W \in X$. Firstly, let us prove that the map is well defined, i.e. that for all $g \in G$, Φ_g is a bijection on X. Indeed, as g is an automorphism, there exists $g' \in G$ such that $g \circ g' = g' \circ g = e_G$, where e_G is the neutral element of G, i.e. $e_G = id_V$. We can then check that $\Phi_g \circ \Phi_{g'} = \Phi_{g'} \circ \Phi_g = id_X$ to conclude that Φ_g is a bijection. It is straightforward to check that $\Phi_{id_V} = \mathrm{id}_X$. Similarly, $\forall g, g' \in G$ and $\forall W \in X$, $(g \circ g')(W) = g(g'(W))$ by definition of compositions of maps, which gives us $\Phi_{gg'} = \Phi_g \circ \Phi_{g'}$. We conclude that Φ is an action

(2) Let us prove that for any $W, W' \in X$, there exists $g \in G$ such that $g(W) = W'$ to conclude that the action is transitive. Consider $\{v_1, v_2\}$ a basis for W that we complete into a basis $\{v_1, v_2, v_3\}$ for V. Similarly, let $\{w_1, w_2\}$ a basis for W' that we complete into a basis $\{w_1, w_2, w_3\}$ for V. The linear map defined by $g : V \to V$, $v_1 \mapsto w_1$, $v_2 \mapsto w_2$, $v_3 \mapsto w_3$ is surjective and thus bijective (so an isomorphism) by the kernel-image theorem. It is clear by its definition that $g(W) = W'$, so we are done.

(3) By point (2), $\forall W \in X, |O_W| = |X|$. Fix a such W. We know that $|O_W| = \frac{|G|}{|Stab_G(W)|}$. Moreover, we know that G is the subgroup of invertible matrices of $M_3(\mathbb{F}_2)$ and thus that $|G| = (2^3-1)(2^3-2)(2^3-4) = 7 \cdot 6 \cdot 4 = 168$ (you have seen this in linear algebra). Let us now find $|Stab_G(W)|$ to conclude. Consider $\{v_1, v_2\}$ a basis for W that we complete into a basis $\{v_1, v_2, v_3\}$ for V. If $g \in G$ fixes W, we must have $g(v_1), g(v_2) \in W$ linearly independent. There are 6 such choices. Now, the choices of defining such a linear isomorphism g when $g(v_1)$ and $g(v_2)$ are fixed come down to the choices of the image of v_3 such that it is not contained in W. There are $|V| - |W| = 4$ of these, so we conclude that $|Stab_G(W)| = 6 \cdot 4 = 24$ and that $|X| = \frac{168}{24} = 7$.

Exercise 5. (1) The orbit-stabiliser theorem gives:

$$|G \cdot 1||\mathrm{Stab}_G(1)| = |G|.$$

Since the action is transitive, there is only one orbit of the action and hence $|G \cdot 1| = n$. Therefore $n \mid |G|$.

(2) We leave it to the reader to check that conjugation defines an action on the set of subgroups of G.

We have that

$$\mathrm{Stab}_G(H) = \{g \in G \,|\, gHg^{-1} = H\}.$$

The above group $\mathrm{Stab}_G(H)$ is known as the normalizer of H in G, often denoted by $N_G(H)$. Note that $N_G(H)$ is the largest subgroup of G which contains H as a normal subgroup.

Exercise 6. We proceed by contradiction. Suppose that for all $x \in X$, there exists $g \in G$ such that $g \cdot x \neq x$. This precisely mean that all stabilizers $\mathrm{Stab}_G(x)$ are strict subgroups of G, and thus have cardinality $|\mathrm{Stab}_G(x)| = p^{k_x} < p^n$ for some $0 \leq k_x < n$. Let $\tilde{X}$ be a set of representatives of the orbits. By a formula seen in class we have:

$$\begin{aligned}|X| &= \sum_{x\in\tilde{X}} \frac{|G|}{|\mathrm{Stab}_G(x)|} \\ &= \sum_{x\in\tilde{X}} \frac{p^n}{p^{k_x}} \\ &= p \cdot \sum_{x\in\tilde{X}} \frac{p^{n-1}}{p^{k_x}}\end{aligned}$$

which proves that $p||X|$. This contradicts the hypothesis and thus there must exists $x \in X$ such that its stabilizer is G, i.e. such that $g \cdot x = x$ for all $g \in G$.